INSCRIBED SEQUENCES OF SURFACES ASSOCIATED WITH GENERALIZED SEQUENCES OF LAPLACE*

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1. Introduction

The theory of conjugate nets of curves on surfaces in a projective space of n dimensions was generalized by Bompiani† in his theory of systems of curves in conjugacy of type ν. Bompiani also generalized the theory of families of asymptotic curves on surfaces by a theory of families of curves in autoconjugacy of type ν.

A set of transformations for surfaces bearing systems of curves in conjugacy of type ν was offered by B. Segre‡, who also gave a system of transformations for surfaces bearing families of curves in autoconjugacy of type $\nu(\nu>1)$. These transformations produce sequences of surfaces quite analogous to the classical sequences of Laplace. In fact the sequences of surfaces bearing systems of curves in conjugacy of type ν are generalizations of the classical sequences of Laplace. It is these generalized sequences to which we refer in the title.

It is the purpose of this paper to point out a large class of sequences associated with any given sequence of surfaces, and to examine the transformations which generate certain special sequences in that class. Several of the special sequences which we study in that class are associated with the above mentioned sequences of Segre.

In §2, we state the geometric basis for a class of sequences called *associated sequences*, and define *generalized inscribed sequences*. These generalized inscribed sequences form a subclass of the above associated sequences. A generalized inscribed sequence is generated by the same kind of transformation as generates the sequence in which it is inscribed. The existence of a large class of the general inscribed sequences is established in §3, and a *web of inscribed sequences* is defined. The existence theorem of §3 is applied in §4 where a study is made of two classes of sequences of surfaces which are inscribed in

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† E. Bompiani, *Sistemi coniugati sulle superficie degli iperspazi*, Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), p. 91.

‡ B. Segre, *Les systèmes conjugués et autoconjugués d'espèce v et leur transformation de Laplace*, Annales Scientifiques de l'Ecole Normale Supérieure, (3), vol. 44 (1927), pp. 153–212.

a given *hyperbolic sequence of Segre*. Two additional types of sequences, inscribed in a given *parabolic sequence of Segre*, are considered in §5.

As a result of this investigation, the transformations of Laplace and Segre are made available for a much less restricted class of surfaces than the class to which they have been applied heretofore.

In the analytical considerations which follow, a point of a surface in a projective space of n dimensions is represented by $n+1$ coordinates x_i, denoted by the single symbol x. The x_i are functions of the curvilinear coordinates u and v, and they have as many partial derivatives with respect to u and v as are needed. Partial derivatives are denoted in accordance with the formula

$$\frac{\partial^{i+j}x}{\partial u^i \partial v^j} = x^{ij}. \tag{1.1}$$

2. Inscribed sequences in general

Consider any sequence T of surfaces

$$\Sigma, \Sigma_1, \Sigma_2, \cdots, \Sigma_i, \cdots,$$

which is generated by repeated application of a definite transformation such that any surface Σ_{i+1} is a transform of the surface Σ_i by means of a one-to-one point correspondence. The surfaces of T have in common all of the properties of the initial surface Σ which are invariant under the generating transformation. In this sense, they will be called *mathematically equivalent* surfaces. Let ω_r be any osculant of the surface Σ_r at a general point P_r, or let ω_r be any osculant at P_r of a curve belonging to a family which lies on the surface Σ_r. In either case, let ω_{r+1} be the corresponding osculant pertaining to the point P_{r+1} of the surface Σ_{r+1}. Associated with the ∞^2 points of Σ_r, there are ∞^2 osculants ω_r, and we shall denote the doubly infinite set by Ω_r. Obviously, there is associated with the sequence of surfaces Σ_i a sequence of sets of osculants Ω_i.

Consider a surface Σ_r', whose points are in a one-to-one correspondence with the ∞^2 osculants ω_r, pertaining to the surface Σ_r, in such a manner that each point of the surface Σ_r' is in united position with its corresponding osculant ω_r. Let Σ_{r+1}' be a similarly described surface whose points are in a one-to-one united correspondence with the osculants ω_{r+1} of the set Ω_{r+1}. The surface Σ_{r+1}' is a transform of the surface Σ_r', by means of the indirect correspondence which connects them. The surfaces $\cdots, \Sigma_r', \Sigma_{r+1}', \cdots$ form a sequence T' of surfaces which will be referred to as an *associated sequence* of the sequence T.

DEFINITION 2.1. *If the points of a surface are in a one-to-one correspondence with a set of* ∞^2 *linear spaces of* ν *dimensions, in such a manner that each point of the surface is in united position with its corresponding linear space, then and only then will the surface be said to be transversal to the set of linear spaces.*

DEFINITION 2.2. *Let* T *denote a sequence of surfaces in which the points of each surface* Σ_i *are joined in a one-to-one manner to the corresponding points of the adjacent surface* Σ_{i+1} *by a set* Ω *of* ∞^2 *linear spaces* ω *of* ν *dimensions, where the spaces* ω *are osculating spaces at points of the surface* Σ_i *to the curves of a family, which lies on the surface* Σ_i, *or the spaces* ω *are osculants of the surface* Σ_i *itself. Let* T' *denote a sequence in which consecutive surfaces are connected in the same manner as those of the sequence* T. *Let the sequence* T' *be related to the sequence* T *so that each surface* Σ_i' *of the sequence* T' *is transversal to the set* Ω_i *of osculants which connect the points of the surfaces* Σ_i *and* Σ_{i+1} *of* T. *Under these conditions the sequence* T' *will be said to be inscribed in the sequence* T. *The sequence* T *will be said to circumscribe the sequence* T'.

The surface Σ_i' of the above inscribed sequence T' may belong to a more general class of surfaces than does the surface Σ_i of the sequence T. For this reason, the point differential equations which represent the definition of the surface Σ_i' will, in general, be of a higher order than the differential equations which represent the definition of the surface Σ_i of the sequence T. However, the surfaces of the two sequences are connected by correspondences of the same kind. The analytical forms of the transformations in the two sequences will be the same. By proving the existence of these generalized inscribed sequences associated with known sequences, we extend the application of known transformations to a more general class of surfaces and to their point differential equations.

It is observed that a classical inscribed sequence of Laplace furnishes a special example under Definition 2.2. The following article will establish the existence of inscribed sequences of great generality.

3. A WEB OF INSCRIBED SEQUENCES

Before defining a web of inscribed sequences, we shall establish the following basic

THEOREM 3.1. *Let* T *denote a sequence of surfaces in which the points of each surface* Σ_{i+1} *are joined in a one-to-one manner to the corresponding points of the preceding surface* Σ_i, *by a set* Ω_i *of* ∞^2 *osculating spaces of* ν *dimensions belonging to the curves of a family on the surface* Σ_i. *Let* Σ_r' *be any surface which is transversal to the set* Ω_r *of osculating spaces* ω_r. *Then it follows that the transversal surface* Σ_r' *belongs to a sequence* T' *of surfaces which is inscribed in the given sequence* T.

Consider a surface Σ, and on it a family F of curves. Let λ denote the curve of the family which passes through the generating point P of the surface Σ. Denote by ω the osculating space of ν dimensions at the point P to the curve λ. Let Σ' and Σ'' denote two surfaces which are transversal to the set Ω of ∞^2 osculating spaces ω pertaining to the ∞^2 points of the curves in the family F. Let P' and P'' denote the points of intersection of the surfaces Σ' and Σ'' respectively with the osculating space ω. As the point P moves along the curve λ on the surface Σ, the points P' and P'' generate two curves, λ' and λ'' respectively, on the surfaces Σ' and Σ''. Denote by ω' and ω'' the osculating spaces of ν dimensions to the curves λ' and λ'' at the respective points P' and P''. We shall show that the osculants ω' and ω'' intersect in a point.

Let $x(u, v)$ be the coordinates of the generating point of the above surface Σ. Let the curves of the above family F be chosen as the parametric u-curves. Choose any other family of curves as the v-curves. The osculating space of ν dimensions ω to the u-curve at the point P is determined by $\nu+1$ points, whose coordinates are x and the first ν derivatives of x with respect to u. Since the generating point P' of the surface Σ' is in contact with the osculant ω, the coordinates y of the point P' can be expressed as

$$y = \sum_{i=0}^{\nu} \alpha_{i0} x^{i0}. \tag{3.1}$$

For a similar reason, the coordinates z of the generating point P'' of the surface Σ'' are

$$z = \sum_{i=0}^{\nu} \beta_{i0} x^{i0}. \tag{3.2}$$

The osculating spaces of ν dimensions ω' and ω'' to the u-curves at P' and P'' are determined by two sets of points, whose coordinates are y and the first ν derivatives of y, and z with the first ν derivatives of z with respect to u. We exhibit these as follows:

$$y^{i0} = \sum_{j=0}^{i+\nu} \alpha_{j0}^{(i0)} x^{j0} \qquad (i = 0, 1, 2, \cdots, \nu), \tag{3.3}$$

$$z^{i0} = \sum_{j=0}^{i+\nu} \beta_{j0}^{(i0)} x^{j0} \qquad (i = 0, 1, 2, \cdots, \nu). \tag{3.4}$$

The $2\nu+2$ coordinates y^{i0} and z^{i0}, on the left of equations (3.3) and (3.4), are expressed linearly in terms of the $2\nu+1$ functions x^{j0}. Hence there exists a linear relation among the z^{i0} and the y^{i0}. That relation will be indicated as

$$\sum_{j=0}^{\nu} \theta_{j0} y^{j0} = \sum_{j=0}^{\nu} \phi_{j0} z^{j0}. \tag{3.5}$$

Equation (3.5) indicates that the osculating space ω' intersects the osculating space ω'' in a point P_1', the coordinates of which are given by either the right or left member of (3.5). These coordinates are denoted as

$$y_1 = \sum_{j=0}^{\nu} \phi_{j0} z^{j0}. \tag{3.6}$$

The surface generated by the point P_1' will be denoted by Σ_1'.

The above facts justify

LEMMA 3.1. *Let ω be the osculating space of ν dimensions at the point P to a curve λ of a family of curves on a surface Σ. Let Σ' and Σ'' be two surfaces which are transversal to the set Ω of ∞^2 osculants ω, pertaining to the ∞^2 points of the surface Σ. Let λ' and λ'' be the curves on Σ' and Σ'' respectively which correspond to the curve λ on Σ. Then it follows that the osculating space of ν dimensions ω' to the curve λ' at a point P' of the surface Σ', and the osculating space of ν dimensions ω'' to the curve λ'' at the point P'' of the surface Σ'', intersect in a point P_1' which generates a surface Σ_1'.*

From this lemma, Theorem 3.1 can be obtained directly by assuming that the surface Σ'' of the lemma is a transform Σ_1 of the surface Σ, and that the surfaces Σ and Σ_1 belong to a sequence of mathematically equivalent surfaces

$$\Sigma, \Sigma_1, \Sigma_2, \cdots, \Sigma_i, \cdots. \tag{3.7}$$

Since the surface Σ_1' bears the same relation to the assumed surface Σ_1 as the surface Σ' bears to the surface Σ, and since the surfaces Σ and Σ_1 are mathematically equivalent, it follows that the surfaces Σ' and Σ_1' are also mathematically equivalent. That is, the surface Σ_1' is a transform of the surface Σ'. Also, since the surface Σ_1 is transformable into the surface Σ_2 of the sequence (3.7), by a repetition of the above argument, it follows that the surface Σ_1' is transformable in the same manner into a surface Σ_2' of the sequence

$$\Sigma', \Sigma_1', \Sigma_2', \cdots, \Sigma_i', \cdots. \tag{3.8}$$

If in equations (3.2), (3.5) and (3.6) we replace the coordinates z by x_1 of the point P_1 of the surface Σ_1, we obtain the following relations:

$$x_1 = \sum_{i=0}^{\nu} \beta_{i0} x^{i0}, \tag{3.9}$$

$$\sum_{j=0}^{\nu} \theta_{j0} y^{j0} = \sum_{j=0}^{\nu} \phi_{j0} x_1^{j0}, \tag{3.10}$$

$$y_1 = \sum_{j=0}^{\nu} \theta_{j0}\, y^{j0}, \tag{3.11}$$

$$y_1 = \sum_{j=0}^{\nu} \phi_{j0}\, x_1^{j0}. \tag{3.12}$$

Equation (3.12) shows that the surface Σ_1', of point coordinates y_1, is transversal to the ∞^2 osculating spaces of ν dimensions to the u-curves of the surface Σ_1. This fact indicates that the sequence (3.8) is inscribed in the sequence (3.7). Equation (3.11) shows that corresponding points of the two surfaces Σ' and Σ_1', of the sequence (3.8), are joined in a one-to-one manner by the osculating spaces of ν dimensions of the u-curves on the surface Σ'. These facts complete the proof of the theorem.

Equation (3.11) shows that the transformation which generates the inscribed sequence (3.8) is of the same analytical form as the transformation (3.9) which generates the circumscribed sequence (3.7).

The inscribed sequence T' of Theorem 3.1 has all of the properties of T which are required by the hypothesis. As a consequence, the theorem is applicable to the sequence T', and repeatedly, showing that there is in general an endless aggregate of sequences of surfaces

$$T',\ T'',\ T''',\ \cdots, \tag{3.13}$$

successively inscribed in a given sequence T.

DEFINITION. *An aggregate of successively inscribed sequences of the type* (3.13) *will be called a web of inscribed sequences. The sequence T will be said to be circumscribed about the web.*

In a given web, the properties of the surfaces vary from one sequence to the next, but the transformations have the same form for the entire web.

4. SEQUENCES INSCRIBED IN A HYPERBOLIC SEQUENCE OF SEGRE

A surface Σ bearing a system of curves in conjugacy of type ν may be defined as an integral surface of a hyperbolic differential equation*

$$\sum_{i=0}^{\nu} \sum_{j=0}^{1} A_{ij}\, x^{ij} = 0. \tag{4.1}$$

Segre's transformation of the First Kind† for the above surface Σ has the form

$$x_1 = \sum_{i=0}^{\nu} b_{i0}\, x^{i0}, \tag{4.2}$$

* B. Segre, loc. cit., p. 161.

† B. Segre, loc. cit., p. 169.

for which the coefficients b_{i0} are determined, to within a proportionality factor, in terms of the A_{ij} of equation (4.1). It is evident from equation (4.2) that the surface Σ_1, generated by the point having coordinates x_1, is transversal to the osculating spaces of ν dimensions to the u-curves on the surface Σ. Corresponding points of the two surfaces Σ and Σ_1 are joined in a one-to-one manner by the osculating spaces of ν dimensions of the u-curves of the surface Σ. By repeated application of the transformation (4.2), a sequence T_+ of surfaces

$$\Sigma, \Sigma_1, \Sigma_2, \cdots \tag{4.3}$$

is generated.

Since each surface of the sequence (4.3) is an integral surface of a hyperbolic differential equation of the type (4.1), we shall refer to the sequence as a *hyperbolic sequence of Segre*. That part of the entire sequence which is generated by the Segre transformation of the First Kind will be called the *forward* or *positive branch*.

From the foregoing remarks, we verify that the positive branch T_+ of the above hyperbolic sequence of Segre has all of the properties required by the hypothesis of Theorem 3.1. Consequently, the theorem is applicable to any surface which is transversal to the ∞^2 osculating spaces which connect corresponding points of any pair of consecutive surfaces in the sequence of Segre. From this fact we have

THEOREM 4.1. *The positive or forward branch T_+ of a hyperbolic sequence of Segre is circumscribed about a web of inscribed sequences*

$$T_+', T_+'', T_+''', \cdots . \tag{4.4}$$

We now consider a class of sequences of surfaces inscribed in the *inverse* or *negative branch* T_- of a hyperbolic sequence of Segre. The transformation which takes the above surface Σ into its transform Σ_{-1} of the negative or inverse branch of the sequence of Segre* is represented by the equation

$$x_{-1} = \sum_{i=0}^{\nu-1} \sum_{j=0}^{1} C_{ij}\, x^{ij}. \tag{4.5}$$

The x_{-1} are the coordinates of the point P_{-1} which generates the surface Σ_{-1}. The coefficients C_{ij} are determined to within a proportionality factor, in terms of the coefficients A_{ij} of equation (4.1).

From the terms of equation (4.5) we observe that the point P_{-1} lies in the sum-space of $2\nu-1$ dimensions, formed by the osculating space of $\nu-1$ dimensions to the u-curve at the point P of Σ, and by the osculating space of

* B. Segre, loc. cit., p. 183.

$\nu-1$ dimensions to the u-curve through the point $(u, v+\Delta v)$, adjacent to P. We shall denote this sum-space by σ, and likewise the corresponding sum-spaces at the generating points of the surfaces $\Sigma_{-1}, \Sigma_{-2}, \cdots$ by the corresponding symbols $\sigma_{-1}, \sigma_{-2}, \cdots$.

Let Σ' denote a surface which is distinct from the two surfaces Σ and Σ_{-1}, but which is transversal to the ∞^2 osculating spaces σ of the surface Σ. The coordinates y of the generating point P' of the surface Σ' can be expressed as

$$y = \sum_{i=0}^{\nu-1} \sum_{j=0}^{1} g_{ij} x^{ij}, \tag{4.6}$$

in which the g_{ij} are arbitrary except that they are distinct from the C_{ij} of equation (4.5).

On examining the two osculating sum-spaces σ_{-1} and σ' of the surfaces Σ_{-1} and Σ', we find, as will be shown analytically in the next paragraph, that they intersect in a point P'_{-1}. The point P'_{-1} generates a surface Σ'_{-1} which is transversal to the ∞^2 osculating spaces σ_{-1}. It is also transversal to the ∞^2 osculants σ' of the surface Σ'. From the mathematical equivalence of the surfaces Σ and Σ_{-1} and by the fact that the surface Σ'_{-1} bears the same relation to the surface Σ_{-1} as the surface Σ' bears to Σ, it follows that the two surfaces Σ' and Σ'_{-1} are mathematically equivalent. The surface Σ'_{-1} is a transform of the surface Σ', and corresponding points of the two surfaces are joined by the ∞^2 osculants σ' of the u-curves on the surface Σ'. These facts justify

THEOREM 4.2. *Let Σ and Σ_{-1} denote two consecutive surfaces of the inverse or negative branch T_{-1} of a hyperbolic sequence of Segre. Let Σ' be any surface which is distinct from the surfaces Σ and Σ_{-1}, and which is transversal to the ∞^2 osculating sum-spaces σ joining corresponding points of Σ and Σ_{-1}. It follows that the surface Σ' belongs to a sequence T'_{-1} of surfaces which is inscribed in the branch T_{-1} of the given sequence of Segre.*

For the analytical justification of the above theorem, we exhibit the coordinates of the points which determine the osculating sum-spaces σ' and σ_{-1} at the points P' and P_{-1} of the surfaces Σ' and Σ_{-1}. By computing derivatives of equations (4.5) and (4.6), we obtain the desired coordinates as the left members of

$$\begin{aligned} \overset{\lambda\mu}{x}_{-1} &= \sum_{i=0}^{\nu-1+\lambda} \sum_{j=0}^{1+\mu} C_{ij}^{(\lambda\mu)} \overset{ij}{x} \qquad (\lambda = 0, 1, \cdots, \nu-1; \mu = 0, 1), \\ \overset{\lambda\mu}{y} &= \sum_{i=0}^{\nu-1+\lambda} \sum_{j=0}^{1+\mu} G_{ij}^{(\lambda\mu)} \overset{ij}{x}. \end{aligned} \tag{4.7}$$

The left members of (4.7) are 4ν functions expressed linearly in terms of the $6\nu-3$ functions x^{ij} $(i=0, 1, \cdots, 2\nu-2; j=0, 1, 2)$. By computing higher derivatives of equation (4.1), it is easily shown that there are $2\nu-2$ linear relations among the above functions x^{ij}. These relations are expressed by the equation

$$\sum_{i=0}^{\nu+\gamma}\sum_{j=0}^{1+\delta} A_{ij}^{(\gamma\delta)} x^{ij} = 0 \quad (\gamma = 0, 1, \cdots, \nu-2; \delta = 0, 1).$$

Hence the above mentioned 4ν functions of the left members of (4.7) are ultimately expressed in terms of $4\nu-1$ of the $6\nu-3$ variables x^{ij}. The left members of (4.5) therefore satisfy a linear relation of the form

$$\sum_{i=0}^{\nu-1}\sum_{j=0}^{1} \theta_{ij} y^{ij} = \sum_{i=0}^{\nu-1}\sum_{j=0}^{1} \psi_{ij} x_{-1}^{ij}. \tag{4.8}$$

This equation indicates that the osculating sum-space σ' of the surface Σ' intersects the osculating sum-space σ_{-1} of Σ_{-1} in the point P'_{-1}, the coordinates y_{-1} of which are obtained from the left member of (4.6) as

$$y_{-1} = \sum_{i=0}^{\nu-1}\sum_{j=0}^{1} \theta_{ij} y^{ij}. \tag{4.9}$$

Equation (4.9) exhibits the analytical form of the transformation which generates the inscribed sequence T'_{-1} indicated by the above theorem. This transformation is essentially of the same form as the transformation (4.5) which generates the negative branch T_{-1} of the hyperbolic sequence of Segre.

5. Sequences inscribed in a parabolic sequence of Segre

A surface Σ bearing a family of curves in autoconjugacy of type ν may be defined as an integral surface of a parabolic differential equation of the type

$$\sum_{i=0}^{\nu+1} A_{i0} x^{i0} + \sum_{i=0}^{\nu-1} A_{i1} x^{i1} = 0.* \tag{5.1}$$

The transformation of Segre†, which takes the above surface Σ into the surface Σ_1 of the positive branch T_+ of a sequence, can be represented by the equation

$$x_1 = \sum_{i=0}^{\nu-1} b_{i0} x^{i0} \qquad (\nu > 1). \tag{5.2}$$

* B. Segre, loc. cit., p. 159.
† B. Segre, loc. cit., p. 206.

The b_{i0} are determined, to within a factor, in terms of the coefficients of equation (5.1).

A Segre sequence of surfaces bearing families of curves in autoconjugacy of type ν will be referred to as a *parabolic sequence of Segre.*

Equation (5.2) shows that in the positive branch of a parabolic sequence of Segre the corresponding points of two adjacent surfaces Σ and Σ_1 are joined, in a one-to-one manner, by the ∞^2 osculating spaces of $\nu-1$ dimensions to u-curves of the surface Σ. On replacing the index ν by $\nu-1$ in Theorem 3.1, we have the resulting

THEOREM 5.1. *The positive branch T_+ of a parabolic sequence of Segre is circumscribed about a web of inscribed sequences of surfaces.*

Attention will now be given to a class of sequences inscribed in the inverse or negative branch T_- of a given parabolic sequence of Segre.

The transformation which carries a given surface Σ of a parabolic sequence of Segre* into its transform Σ_{-1} of the negative branch has the analytic form

$$(5.3) \qquad x_{-1} = \sum_{i=0}^{\nu} C_{i0}\, x^{i0} + \sum_{i=0}^{\nu-2} C_{i1}\, x^{i1},$$

in which the C_{ij} are uniquely defined, except for a proportionality factor, in terms of the A_{ij} of (5.1).

Equation (5.3) shows that the point P_{-1}, which generates the surface Σ_{-1}, is in the sum-space formed by the osculating space of ν dimensions to the u-curve through the point P of Σ and by the osculating space of $\nu-2$ dimensions to the u-curve through the point $(u, v+\Delta v)$ of Σ. We shall denote this osculating sum-space by σ, and shall denote the corresponding sum-space of the surface Σ_{-1} by σ_{-1}, etc.

Let Σ' represent any surface, distinct from the surfaces Σ and Σ_{-1}, which is transversal to the osculating sum-spaces σ of the surface Σ. On examining the osculating sum-spaces σ' and σ_{-1} of the surfaces Σ' and Σ_{-1} we find, as will be demonstrated analytically later, that these two sum-spaces intersect in a point P'_{-1}. The point P'_{-1} generates a surface Σ'_{-1} which is transversal to the osculating sum-spaces σ_{-1} and σ' of the surfaces Σ_{-1} and Σ'. The surface Σ'_{-1} bears the same relation to the surface Σ_{-1} as the surface Σ' bears to the surface Σ. Since the surfaces Σ and Σ_{-1} are mathematically equivalent, it follows that the surfaces Σ' and Σ'_{-1} are mathematically equivalent. The surface Σ'_{-1} is a transform of the surface Σ', and corresponding points of the

* B. Segre, loc. cit., p. 209.

two surfaces are joined by the osculating sum-spaces σ' at points of the surface Σ'. From these facts we have

THEOREM 5.2. *Let Σ and Σ_{-1} be any two consecutive surfaces in the inverse or negative branch T_- of a parabolic sequence of Segre. Let Σ' be any third surface which is transversal to the ∞^2 connecting sum-spaces σ pertaining to the surface Σ. Then it follows that the surface Σ' belongs to a branch T'_{-1} of a sequence of surfaces inscribed in the given sequence of Segre.*

To justify the above theorem analytically we exhibit the coordinates of the points which determine the osculating sum-space σ_{-1} at the point P_{-1} of the surface Σ_{-1}. By taking derivatives of the x_{-1}, as expressed in (5.3), we obtain

$$(5.4)\quad \begin{aligned} \overset{\lambda 0}{x}_{-1} &= \sum_{i=0}^{\nu+\lambda} C_{i0}^{(\lambda 0)} x^{i0} + \sum_{i=0}^{\nu-2+\lambda} C_{i1}^{(\lambda 0)} \overset{i1}{x} && (\lambda = 0, 1, \cdots, \nu),\\ \overset{\mu 1}{x}_{-1} &= \sum_{i=0}^{\nu+\mu}\sum_{j=0}^{1} g_{ij}^{(\mu 1)} \overset{ij}{x} + \sum_{i=0}^{\nu-2+\lambda}\sum_{j=0}^{2} h_{ij}^{(\mu 1)} \overset{ij}{x} && (\mu = 0, 1, \cdots, \nu-2). \end{aligned}$$

In a similar manner the coordinates y of the point P' and the remaining points which determine the sum-space σ' at the point P' of the surface Σ' can be displayed in the form

$$(5.5)\quad \begin{aligned} \overset{\lambda 0}{y} &= \sum_{i=0}^{\nu+\lambda} M_{i0}^{(\lambda 0)} \overset{i0}{x} + \sum_{i=0}^{\nu-2+\lambda} M_{i1}^{(\lambda 0)} \overset{i1}{x} && (\lambda = 0, 1, \cdots, \nu),\\ \overset{\mu 1}{y} &= \sum_{i=0}^{\nu+\mu}\sum_{j=0}^{1} K_{ij}^{(\mu 1)} \overset{ij}{x} + \sum_{i=0}^{\nu-2+\lambda}\sum_{j=0}^{2} P_{ij}^{(\mu 1)} \overset{ij}{x} && (\mu = 0, 1, \cdots, \nu-2). \end{aligned}$$

The left members of (5.4) and (5.5) are 4ν functions expressed linearly in terms of the $6\nu-3$ functions $x^{00}, x^{10}, \cdots, x^{2\nu,0}; x^{01}, x^{11}, \cdots, x^{2\nu-2,1}; x^{02}, \cdots, x^{2\nu-4,2}$. But by means of equation (5.1) and its derivatives we have $2\nu-2$ linear relations among the above $6\nu-3$ functions. We exhibit the $2\nu-2$ relations as follows:

$$(5.6)\quad \begin{aligned} &\sum_{i=0}^{\nu+1+\lambda} A_{i0}^{(\lambda 0)} \overset{i0}{x} + \sum_{i=0}^{\nu-1+\lambda} A_{i1}^{(\lambda 0)} \overset{i1}{x} = 0 && (\lambda = 0, 1, \cdots, \nu-1),\\ &\sum_{i=0}^{\nu+1+\mu}\sum_{j=0}^{1} B_{ij}^{(\mu 1)} \overset{ij}{x} + \sum_{i=0}^{\nu-1+\mu}\sum_{j=1}^{2} D_{ij}^{(\mu 1)} \overset{ij}{x} = 0 && (\mu = 0, 1, \cdots, \nu-3). \end{aligned}$$

By means of the $2\nu-2$ relations (5.6), the 4ν left members of equations (5.4) and (5.5) are expressed linearly in terms of $4\nu-1$ derivatives of x. Hence the left members of (5.4) and (5.5) satisfy a linear relation of the form

$$\sum_{i=0}^{\nu}\theta_{i0}\, y^{i0}+\sum_{i=0}^{\nu-2}\theta_{i1}\, y^{i1}=\sum_{i=0}^{\nu}\phi_{i0}\, x_{-1}^{i0}+\sum_{i=0}^{\nu-2}\phi_{i1}\, x_{-1}^{i1}. \tag{5.7}$$

This equation shows that the osculating sum-spaces σ' and σ_{-1} meet in a point P_{-1} the coordinates y_{-1} of which are given by the left members, which we exhibit as

$$y_{-1}=\sum_{i=0}^{\nu}\theta_{i0}\, y^{i0}+\sum_{i=0}^{\nu-2}\theta_{i1}\, y^{i1}. \tag{5.8}$$

The transformation (5.8), which sends the surface Σ' into the surface Σ'_{-1}, is obviously of the same form as the transformation (5.3) which sends the surface Σ into the surface Σ_{-1}.

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